Small Ranks of Central Unit Groups of Integral Group Rings of Alternating Groups

R. Zh. Aleev

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Abstract—We prove that the ranks of central unit groups of integral group rings of alternating groups of degrees greater than 38 are at least 11. The presented tables contain the ranks of all central unit groups of integral group rings of alternating groups of degrees at most 200. In particular, for every \( r \in \{0, \ldots, 10\} \), we obtain the complete list of integers \( n \) such that the central unit group of the integral group ring of the alternating group of degree \( n \) has rank \( r \).

Keywords: alternating group, group ring, central unit, rank of abelian group, partition.

DOI:

On V.D. Mazurov and A.A. Makhnev’s jubilees

INTRODUCTION

Frobenius showed [4] how a partition of a positive integer \( n \) can be used to construct an irreducible complex character of the symmetric group \( S_n \). Studying irreducible characters of alternating groups, he also related them to certain partitions and specified all nonintegral values of irreducible characters.

The degrees of alternating groups in which central unit groups of integral group rings have rank 0 were found in [5].

In [1] the ranks of central unit groups of integral group rings in alternating groups of degrees up to 36 were found and it was proved that, for any \( n \geq 36 \), the central unit group of the group \( A_n \) has rank at least 2. Kargapolov [2, 3] calculated the ranks of central unit groups of integral group rings for alternating groups of degrees not exceeding 800.

The main aim of this paper is to find integers \( n \) for which the rank of the central unit group of the integral group ring of an alternating group of degree \( n \) does not exceed 10. Our approach to this study is different from the approaches of the mentioned papers.

In what follows, we will use the following notation and definitions.

We denote by \( r_n \) the rank of the central unit group of the integral group ring of the alternating group of degree \( n \).

A partition of a positive integer \( n = a_1 + \ldots + a_k \), where \( a_i \in \mathbb{N} \), will be denoted by \([a_1, \ldots, a_k]\).

An integer will be called a square (a nonsquare) if it is (is not) the square of a positive integer.

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1. PRELIMINARIES AND FORMULATION OF THE MAIN RESULTS

Lemma 1 [5, Theorem 4.5]. The rank $r_n$ equals the number of partitions $a = [a_1, \ldots, a_k]$ of the positive integer $n$ that satisfy the following conditions:

1. $a_i$ is odd for $1 \leq i \leq k$;
2. $a_i \neq a_j$ for $i \neq j$;
3. $n \equiv k \pmod{4}$;
4. $\prod_{i=1}^{k} a_i$ is not a square.

Denote by $F_n$ the set of all partitions of the positive integer $n$ that satisfy conditions (1)–(4) of Lemma 1.

The author found the ranks $r_n$ for $n \leq 200$ as early as 20 years ago. Later these results were verified and recalculated many times by the author’s students but were never published. The tables presented below were obtained by Kargapolov (they were not published either), for which the author is very grateful.

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$n \equiv 1 \pmod{8}$

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\[ n \equiv 8 \pmod{8} \]

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The following statement is extremely important and useful.

**Lemma 2** [4, pp. 179, 188]. Let \( \alpha \) be some partition of an integer \( n \) satisfying the first three conditions of Lemma 1. Then:

1. The number of elements of the partition \( \alpha \) does not exceed \( \lfloor \sqrt{n} \rfloor \). In particular, if \( k_n \) is the maximum possible number of elements of such a partition \( \alpha \) of the integer \( n \), then
   
   - For \( n \equiv 1 \pmod{4} \), \( k_n \geq 5 \iff n \geq 25 \);
   - For \( n \equiv 2 \pmod{4} \), \( k_n \geq 6 \iff n \geq 38 \);
   - For \( n \equiv 3 \pmod{4} \), \( k_n \geq 7 \iff n \geq 51 \);
   - For \( n \equiv 4 \pmod{4} \), \( k_n \geq 4 \iff n \geq 16 \).

2. The product of elements of the partition \( \alpha \) is congruent to 1 modulo 4.

In this paper we prove the following theorem.

**Theorem.** (1) If \( n \leq 38 \), then the results presented in the above tables are valid.
(2) If \( n \geq 39 \), then \( r_n \geq 11 \).

### 2. PROOF OF THE THEOREM

Let us prove the first statement of the theorem.

Since this statement was proved for \( n \leq 36 \) in [1] (the above tables also confirm this), we will consider only the cases \( n = 37 \) and \( n = 38 \).

Let us write all partitions of the numbers \( n = 37 \) and \( n = 38 \) that satisfy the first three conditions of Lemma 1. Let \( k_\alpha \) be the number of elements of a partition \( \alpha \) of \( n \in \{37, 38\} \) that satisfies the first three conditions of Lemma 1. Then, by Lemma 2, we have \( k_\alpha \in \{1, 5\} \) for \( n = 37 \) and \( k_\alpha \in \{2, 6\} \) for \( n = 38 \). Thus, we have partitions

\[
\begin{align*}
[37], & \quad [1, 3, 5, 7, 21], \quad [1, 3, 5, 9, 19], \quad [1, 3, 5, 11, 17], \quad [1, 3, 5, 13, 15], \quad [1, 3, 7, 9, 17], \\
[1, 3, 7, 11, 15], & \quad [1, 3, 9, 11, 13], \quad [1, 5, 7, 9, 15], \quad [1, 5, 7, 11, 13], \quad \text{and} \quad [3, 5, 7, 9, 13]
\end{align*}
\]

for \( n = 37 \) and partitions

\[
\begin{align*}
[1, 37], & \quad [3, 35], \quad [5, 33], \quad [7, 31], \quad [9, 29], \quad [11, 27], \quad [13, 25], \quad [15, 23], \quad [17, 21], \quad \text{and} \quad [1, 3, 5, 7, 9, 13]
\end{align*}
\]

for \( n = 38 \). Since the product of elements of each partition is not a square, we have \( r_{37} = 11 \) and \( r_{38} = 10 \). Statement (1) of the theorem is proved.

Let us prove the second statement of the theorem.

Let \( n \geq 39 \). We must prove that \( r_n \geq 11 \). To do this, it is sufficient to specify for each \( n \geq 39 \) 11 different partitions of \( n \) that satisfy Lemma 1. In the following argument we use the following trivial but useful remark: the square of any odd number is congruent to 1 modulo 8.

We will consider eight cases that correspond to congruences \( n \equiv \varepsilon \pmod{8} \), where \( \varepsilon \in \{1, \ldots, 8\} \).

Let us start with the case \( n \equiv 2 \pmod{8} \). We have \( n \geq 42 \), since we assumed that \( n \geq 39 \). This is the key case in this section, because it is easy to see from the subsequent presentation that the argument of this case can be transferred to other cases.

**Proposition 1.** Let \( n \equiv 2 \pmod{8} \) and \( n \geq 42 \). Then \( r_n \geq 11 \).

**Proof.** The proof will be carried out in three steps.
Step 1. Consider partitions of the form $[k, n - k]$ with odd $k$. Note that, if $k(n - k)$ is a square, then $k \equiv 1 \pmod{4}$. Indeed,

$$k(n - k) \equiv 1 \pmod{8} \iff k(2 - k) \equiv 1 \pmod{8} \iff 2 - k \equiv k \pmod{8} \iff 2k \equiv 2 \pmod{8} \iff k \equiv 1 \pmod{4}.$$ 

Step 2. Let $[k, n - k]$ be a partition with $k = 4k_1 + 3$, and let $n = 8n_1 + 2$ ($k_1, n_1 \in \mathbb{N} \cup \{0\}$). Then, according to Step 1, $(4k_1 + 3)(8n_1 - 4k_1 - 1)$ is not a square for any $k_1 \in \{0, \ldots, \tilde{k}_1\}$, where $\tilde{k}_1$ is the greatest of integers $k_1$ such that $4k_1 + 3 < 8n_1 - 4k_1 - 1$; i.e., $\tilde{k}_1$ is a solution of the following system (with respect to $k_1$):

$$\begin{align*}
4k_1 + 3 \leq 8n_1 - 4k_1 - 1 - 2 & \iff 8k_1 \leq 8n_1 - 6 \\
4(k_1 + 1) + 3 \geq 8n_1 - 4k_1 - 1 & \iff 8k_1 \geq 8n_1 - 8 \iff 2k_1 \geq 2n_1 - \frac{3}{2}.
\end{align*}$$

Since $k_1$ must be integer, we obtain $2k_1 = 2n_1 - 2$, which implies $\tilde{k}_1 = n_1 - 1$. It is clear that

$$\tilde{k}_1 = n_1 - 1 \geq 10 \iff n_1 \geq 11 \iff n = 8n_1 + 2 \geq 88 + 2 = 90.$$ 

Thus, Proposition 1 is valid for $n \geq 90$, and there are at least $\tilde{k}_1 + 1 = n_1 - 1 + 1 = n_1$ partitions in $F_n$ among partitions of the form $[k, n - k]$ in the remaining cases.

Step 3. Thus, consider $n \in \{42, 50, 58, 66, 74, 82\}$ and obtain $n_1 \in \{5, 6, 7, 8, 9, 10\}$ partitions in $F_n$, respectively. The products of elements of the partitions

$$[5, n - 5] \in \{[5, 37], [5, 45], [5, 53], [5, 61], [5, 69], [5, 67]\}$$

are not squares except for $[5, 45]$, which, as a matter of fact, means that the constraint $k \equiv 3 \pmod{4}$ is essential for the condition $[k, n - k] \in F_n$. Hence, the statement of Proposition 1 holds for $n = 82$, and, for $n = 42, 50, 58, 66, 74$, there are at least 6, 6, 8, 9, and 10 partitions in $F_n$, respectively.

For $n \in \{42, 50, 58, 66\}$ the partitions

$$[9, n - 9] \in \{[9, 33], [9, 41], [9, 49], [9, 57], [9, 65]\}$$

belong to $F_n$ except for $[9, 49]$. Hence, the statement of the proposition holds for $n = 74$, and, for $n = 42, 50, 58, 66$, there are at least 7, 7, 8, and 9 partitions in $F_n$, respectively.

Similarly, for $n \in \{42, 50, 58, 66\}$, the partitions

$$[13, n - 13] \in \{[13, 29], [13, 37], [13, 45], [13, 53]\},$$

$$[17, n - 17] \in \{[17, 25], [17, 33], [17, 41], [17, 49]\}$$

lie in $F_n$; hence, $n \neq 66$, and, for $n \in \{42, 50, 58\}$, there are at least 9, 9, and 10 partitions in $F_n$, respectively.

For $n \in \{50, 58\}$, the partitions

$$[21, n - 21] \in \{[21, 29], [21, 37]\}$$

belong to $F_n$; hence, $n \neq 58$, and, for $n \in \{42, 50\}$, there are at least 9 and 10 partitions in $F_n$, respectively.
For \( n = 50 \), there is the six-element partition \([3, 5, 7, 9, 11, 15]\) in \( F_n \) (there are no pairs!); hence, \( n \neq 50 \).

Note that (we will use this fact below) we did not consider partitions containing 1; for \( n = 42 \), this will be inevitable, and we will add to the partitions lying in \( F_{42} \) the pair \([1, 41]\) (there are no more pairs!) and the six-element partition \([1, 3, 5, 7, 9, 17]\).

This completes the proof of Proposition 1.

**Proposition 2.** Let \( n \equiv 6 \pmod{8} \) and \( n \geq 46 \). Then \( r_n \geq 11 \).

**Proof.** The proof will be carried out in three steps, as the proof of Proposition 1.

**Step 1.** Consider once again partitions of the form \([k, n - k]\) with odd \( k \). Note that, if \( k(n - k) \) is a square, then \( k \equiv 3 \pmod{4} \), which is proved as in Step 1 of the proof of Proposition 1.

**Step 2.** Let \([k, n - k]\) be a partition with \( k = 4k_1 + 1 \) and \( n = 8n_1 + 6 \). Then \((4k_1 + 1)(8n_1 - 4k_1 + 5)\) is not a square for any \( k_1 \in \{0, \ldots, \hat{k}_1\} \), where \( \hat{k}_1 \) is the largest integer \( k_1 \) such that \( 4k_1 + 3 < 8n_1 - 4k_1 - 1 \). Then, as in Step 2 of the proof of Proposition 1, we can show that \( \hat{k}_1 = n_1 \) and \( \hat{k}_1 \geq 10 \iff n \geq 86 \).

**Step 3.** The proof is completed as in Step 3 of the proof of Proposition 1.

The proposition is proved.

**Proposition 3.** Let \( n \equiv 3 \pmod{8} \) and \( n \geq 43 \). Then \( r_n \geq 11 \).

**Proof.** Consider triples \([1, k, n - k - 1]\) with odd \( k \). It is clear that \([k, n - k - 1]\) satisfies Step 1 of the proof of Proposition 1, and we can use the result of that step. By the proof of Proposition 1, for any \( n - 1 \geq 50 \) we can find the required number of partitions in \( F_n \) that do not contain 1. Therefore, we should consider only \( n = 43 \). We obtain from the proof of Proposition 1 that there are 9 triples in \( F_n \). We can add \([3, 5, 35]\) and \([3, 7, 33]\); hence, \( r_{43} \geq 11 \), which completes the proof. The proposition is proved.

**Proposition 4.** Let \( n \equiv 7 \pmod{8} \) and \( n \geq 39 \). Then \( r_n \geq 11 \).

**Proof.** The proof is similar to the proof of Proposition 3.

**Proposition 5.** Let \( n \equiv 4 \pmod{8} \) and \( n \geq 44 \). Then \( r_n \geq 11 \).

**Proof.** According to the general strategy, we try to use the preceding cases, more exactly, results from the proof of Proposition 1. Consider quadruples of the form \([1, 9, k, n - k - 10]\). By Step 1 of the proof of Proposition 1, the product of elements of quadruples of the form \([1, 9, 4k_1 + 3, 8n_1 - 4k_1 - 9]\) are nonsquares, and their number (see Step 2 of the proof of Proposition 1) is \( n_1 - 1 \). Therefore, Proposition 5 is valid for \( n_1 \geq 12 \) (equivalently, for \( n = 8n_1 + 4 \geq 100 \)). There remain \( n \in \{44, 52, 60, 68, 76, 84, 92\} \). Removing the elements 1 and 9 from the quadruples, we obtain the pairs \([4k_1 + 3, 8n_1 - 4k_1 - 9]\), which are partitions of the numbers \( 8n_1 - 6 = n - 10 \equiv 2 \pmod{8} \). Consequently, we can partially use the proof of Proposition 1.

Thus, we will follow the proof of Proposition 1. For \( n = 44, 52, 60, 68, 76, 84 \), and 92 we have \( 4, 5, 6, 7, 8, 9, \) and 10, respectively, quadruples of nonsquares. In what follows, it is sufficient to consider only the pairs \([k, n - 10 - k]\) for \( n - 10 \). Adding pairs of the form \([5, n - 15]\), we can omit the case \( n = 92 \) and, as in the proof of Proposition 1, find for \( n \in \{44, 52, 60, 68, 76, 84\} \) that there are \( 5, 6, 8, 9, \) and 10, respectively, partitions in \( F_n \). Since the partitions \([13, n - 23] \in \{[13, 21], [13, 29], [13, 37], [13, 45], [13, 53]\}\) are nonsquares, we have \( n \neq 84 \), and, for \( n \in \{44, 52, 60, 68, 76\} \), there are at least 6, 7, 7, 9, and 10 partitions in \( F_n \).

Now, for \( n \in \{44, 52, 60, 68, 76\} \), the partitions

\([1, 3, 5, n - 9] \in \{[1, 3, 5, 35], [1, 3, 5, 43], [1, 3, 5, 51], [1, 3, 5, 59], [1, 3, 5, 67]\}\),
[1, 3, 7, n − 11] ∈ \{[1, 3, 7, 33], [1, 3, 7, 41], [1, 3, 7, 49], [1, 3, 7, 57], [1, 3, 7, 65]\}

belong to \(F_n\), and, hence, for \(n = 44, 52, 60, 68,\) and 76 we have at least 8, 9, 9, 11, and 12, respectively, partitions in \(F_n\). There remain only \(n \in \{44, 52, 60\}\), for which the partitions

\[
[1, 5, 7, n − 13] \in \{[1, 5, 7, 31], [1, 5, 7, 39], [1, 5, 7, 47]\},
\]

\[
[1, 5, 9, n − 15] \in \{[1, 5, 9, 29], [1, 5, 9, 37], [1, 5, 9, 45]\}
\]

belong to \(F_n\) except for \([1, 5, 9, 45]\); therefore, for \(n = 44, 52,\) and 60, we have at least 10, 11, and 10, respectively, partitions in \(F_n\). Thus, \(n \neq 52,\) and for \(n \in \{44, 60\}\) we add the partitions

\[
[1, 3, 11, n − 15] \in \{[1, 3, 11, 29], [1, 3, 11, 45]\},
\]

which completes the proof of the proposition.

**Proposition 6.** Let \(n \equiv 8 \pmod{8}\) and \(n \geq 40\). Then \(r_n \geq 11\).

**Proof.** The proof is similar to the proof of Proposition 5.

**Proposition 7.** Let \(n \equiv 5 \pmod{8}\) and \(n \geq 45\). Then \(r_n \geq 11\).

**Proof.** The proof will be carried out in three steps, as the proof of Proposition 1.

**Step 1.** Consider once again partitions of the form \([1, 3, 5, k, n − k − 9]\) with odd \(k\) and, as in the proof of Proposition 1, find that \(15k(n − k − 9)\) is a nonsquare.

**Step 2.** Let \(k = 2k_1 + 7\) and \(n = 8n_1 + 5\). Then, five-element partitions of the form \([1, 3, 5, 2k_1 + 7, 8n_1 − 2k_1 − 11]\) are contained in \(F_n\) for any \(k_1 \in \{0, \ldots, \hat{k}_1\}\), where \(\hat{k}_1\) is the largest integer \(k_1\) such that \(2k_1 + 7 < 8n_1 − 2k_1 − 7\). As in Step 2 of the proof of Proposition 1, we find that \(\hat{k}_1 = 2n_1 − 5\) and \(\hat{k}_1 \geq 10 \iff n \geq 69\).

**Step 3.** For \(n \geq 69\) we have found the required number of partitions in \(F_n\), and for the remaining cases \(n \in \{45, 53, 61\}\) we have at least 6, 8, and 10 five-element partitions in \(F_n\), respectively. For \(n \in \{45, 53, 61\}\) the partitions

\[
[1, 3, 7, 9, n − 20] \in \{[1, 3, 7, 9, 25], [1, 3, 7, 9, 33], [1, 3, 7, 7, 41]\},
\]

\[
[1, 3, 7, 11, n − 22] \in \{[1, 3, 7, 11, 23], [1, 3, 7, 11, 31], [1, 3, 7, 11, 39]\},
\]

\[
[1, 3, 7, 13, n − 24] \in \{[1, 3, 7, 13, 21], [1, 3, 7, 13, 29], [1, 3, 7, 13, 37]\},
\]

\[
[1, 3, 7, 15, n − 26] \in \{[1, 3, 7, 15, 19], [1, 3, 7, 15, 27], [1, 3, 7, 15, 35]\}
\]

belong to \(F_n\) except for \([1, 3, 7, 15, 35]\); hence, for \(n = 45, 53,\) and 61 we have at least 10, 12, and 13, respectively, five-element partitions in \(F_n\). Thus, it remains to consider the case \(n = 45\). For this case, we already have 10 partitions from \(F_n\), and we complete the proof by adding \([1, 3, 9, 11, 21]\).

**Proposition 8.** Let \(n \equiv 1 \pmod{8}\) and \(n \geq 41\). Then \(r_n \geq 11\).

**Proof.** The proof will be carried out in three steps, as the proof of Proposition 1.

**Step 1.** Consider once again partitions of the forms \([1, 3, 7, k, n − k − 11]\) and \([1, 3, 11, k, n − k − 15]\) with odd \(k\). As in Step 1 of the proof of Proposition 1, we find that, if \(21k(n − k − 9)\) is a square or \(33k(n − k − 15)\) is a square, then \(k \equiv 1 \pmod{4}\).

**Step 2.** The five-element partitions \([1, 3, 7, 4k_0 + 11, 8n_1 − 1 − 4k_0 − 11 − 11] = [1, 3, 7, 4k_0 + 11, 8n_1 − 4k_0 − 21] and [1, 3, 11, 4k_1 + 15, 8n_1 − 1 − 4k_0 − 15 − 15] = [1, 3, 7, 4k_1 + 11, 8n_1 − 4k_1 − 29]\) always give nonsquares for \(k_0 \in \{0, \ldots, \hat{k}_0\}\), where \(\hat{k}_0\) is the largest integer \(k_0\) such that \(4k_0 + 11 < 8n_1 − 8,\)
$4k_1 - 21$, and $k_1 \in \{0, \ldots , \hat{k}_1\}$, where $\hat{k}_1$ is the largest integer $k_1$ such that $4k_1 + 15 < 8n_1 - 4k_1 - 29$.

As in Step 2 of the proof of Proposition 1, we obtain $\hat{k}_0 = n_1 - 5$ and $\hat{k}_1 = n_1 - 6$.

**Step 3.** The proof is completed similarly to Step 3 of the proof of Proposition 7.

Statement 2 of the theorem follows immediately from Propositions 1–8.

The theorem is proved.

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*Translated by E. Vasil’eva*