Parallel solver for hp spectral discretizations of 3-d elliptic equations*

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As it is well known, two factors are equally important for speeding up large scale computation: the use of algorithms cheap in the respect of the arithmetic cost and the possibility of parallelization of computation. We introduce fast, in other words almost optimal in the arithmetic operations count, parallel domain decomposition preconditioner-solver for spectral finite element *hp*-discretizations of 2-*nd* order elliptic equations in 3-d domains. This result is essentially based on the specific interrelation between the stiffness matrices of the spectral and hierarchical reference p-elements derived by Korneev and Rytov recently. This interrelation allows to apply to the spectral element discretizations fast solvers which are quite similar to those developed earlier for the hierarchical discretizations. We present an algorithm which allows basic parallelization by finite elements and their faces and, if necessary, even deeper parallelization.

1. Introduction

Contemporary practice of scientific computations requires solution of systems of discrete equations with giant numbers of degrees of freedom. In one simulation of an earthquake in the Southern California, the discrete model used the grid with 1.8 billion nodes ($>$ 5 billion unknowns) and produced $>$ 50 terabytes of data of information [9]. Naturally, the success of the simulation in an acceptable time span was provided by the use of a high performance supercomputer, maximal parallelization of computations, allowed by the supercomputer, and high efficiency solving algorithms, *i.e.,* solvers. The prallelization was arranged by means of the domain decomposition method. Although dramatic increases in computer performance are well known, advanced algorithms have contributed as much to increases in computational simulation capability as have improvements in hardware. According to the Moore's Law the power of one computer is doubled each 18-24 months [24]. The authors of [28] compared with the Moore's Law the gains in the performance from the introduction at the corresponding time of improved algorithms for solving linear systems arising from the discretization of partial differential equations. In particular, they considered Seidel's, optimal SOR, conjugate gradient and multigrid iterative and some direct methods. As they found, these gains either track or exceed those from hardware performance improvements from Moore's Law [28, 53-54 p.p.]. Therefore, an efficient simulation on supercomputers can be produced by numerical methods, which satisfy the three conditions: the method provides the highest relation of accuracy to the number of degrees of freedom, the low arithmetical cost and deep parallelization of computations. It is well known that in many cases *hp* finite element discretizations of elliptic second order equations provide the fastest (exponential) convergence and, therefore, satisfy first condition. In this paper we illustrate that the two other conditions can be also satisfied by the application of the DD (domain decomposition) preconditioner-solver, basic features of which were discussed in [20]- [22].

Solution of internal Dirichlet problems on subdomains of decomposition and problems on their faces usually put the main contribution to the overall computational work. In the case of *hp* finite element discretizations, a good choice for subdomains of decomposition is

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the domains of finite elements. For this reason, under the conditions of the shape regularity of finite elements, optimization of these components with respect to the computational work is reduced to obtaining fast preconditioners-solvers for the stiffness matrix of the p reference element and the Schur complement related to its boundary.

In spite of well developed general theory of DD preconditioners for hp–discretizations, providing almost optimal relative condition numbers and a deep parallelization of computations, see [3], [1], [27], [11], [25], [15], [31] and others, optimization (especially in 3-d case) of two main components (the local Dirichlet problems and problems on the faces of finite elements) in respect to the number of arithmetic operations started recently. As a starting point for obtaining such optimized components for the two major types of hp–discrtetizations, there were primarily used the preconditioners of a finite-difference type, suggested in [11], [16] and Orzag [26] for the respective reference element stiffness matrices. On the basis of these preconditioners further steps to the fast preconditioners-solvers were done primarily for the so called hierarchical discretizations. They are generated by the reference element with the form functions, produced by the tensor products of the integrated Legendre's polynomials. For such reference elements, a number of fast preconditioners-solvers for the internal stiffness matrices have been derived, justified theoretically and tested numerically, see for instance [13, 14], [4], [5] and [18, 19]. For the spectral elements, there was known, *e.g.,* the multilevel solver [29], which efficiency was well approved only numerically.

Recently, Korneev and Rytov [20, 21] established some essential interrelation between the hierarchical and spectral cubic reference elements. This interrelation allowed to adapt all solvers, known for reference element of one type, into the solvers for stiffness matrices of other type reference element with the same computational cost. For instance, Korneev/Rytov [20] justified the fast multilevel solver for spectral elements, which is of the same type with one suggested by Beuchler [4] for the 2-d hierarchical p–element.

In this paper, we present fast parallel domain decomposition preconditioner-solver, containing the following components. For the internal Dirichlet problem on the spectral element, the fast preconditioner-solver is based on the multiresolution wavelet preconditioners-solvers for 1-d stiffness and mass matrices, which are similar to those used in [5] in the case of hierarchical reference elements. The set of admissible wavelets satisfies even easier conditions in comparison with the conditions arising for hierarchical elements. The fast preconditioner-solver for problems on faces is designed basically by means of the K-interpolation. Taking into account that inefficient prolongations from the interface boundary can decrease efficiency of the DD algorithm, in the paper we use the prolongations constructed by means of the inexact iterative solver for the inner problems with the pointed out above multiresolution wavelet preconditioner which leads to the almost optimal prolongation operations. The rest component of the DD preconditioner-solver is a good preconditioner for the wire basket subproblem (with a relatively small dimension $\mathcal{O}(\mathcal{R}p)$, where $\mathcal{R}$ is the number of finite elements). We use the preconditioner studied for h discretizations in [30] and [8] and expanded to spectral ones in [27] and [6].

The paper is organized as follows. In section 2, we introduce the spectral finite element discretization. Preconditioners for the spectral reference element are described in section 3. Section 4 is devoted to the DD preconditioner-solver. Its components are presented in the following sections. Multiresolution wavelet preconditioners-solvers for the internal Dirichlet problems for finite elements and their faces are described in sections 5 and 6, correspondingly. Iterative operation of prolongation from the interelement boundary inside the finite elements is introduced in section 7. Wire-basket component is described in section 8, it consists of the wire-basked preconditioner and operator of prolongation onto the interelement boundary. Results concerning the effectiveness of the preconditioner-solver, its computational cost and computational cost of its components are presented in section 9. In this section we also introduce the parallel solver for the DD preconditioner with parallelization by finite elements and their faces.

Let us describe some notations used farther in this paper. The reference cube is denoted by $\tau_0 = (-1,1)^3$. The functional space on the cubic reference elements of all types, considered in the paper, is the space $\mathcal{Q}_{p,\mathbf{x}}$ of the polynomials of order $p \geq 1$ in each variable of $\mathbf{x} = (x_1, x_2, x_3)$. We consider the Lagrange elements with the the nodes of the Gauss-Lobatto-Legendre (GLL) and Gauss-Lobatto-Chebyshev (GLC) quadrature formulas. Signs $\prec$, $\succ$, $\asymp$ are used for the inequalities and equalities held up to positive absolute constants; $\mathbf{A}^+$ – pseudo-inverse of a matrix $\mathbf{A}$; $\mathbf{A} \prec \mathbf{B}$ with nonnegative matrices $\mathbf{A}, \mathbf{B}$ implies $\mathbf{v}^\top \mathbf{A} \mathbf{v} \prec \mathbf{v}^\top \mathbf{B} \mathbf{v}$ for any vector $\mathbf{v}$ and similarly for signs $\succ$, $\asymp$, $<$, $>$; for a symmetric nonnegative matrix $\mathbf{A}$ and $\forall\, \mathbf{v}$ it is assumed $||\mathbf{v}||^2_{\mathbf{A}} := \mathbf{v}^\top \mathbf{A} \mathbf{v}$. Notations $|\cdot|_{k,\Omega}$, $\|\cdot\|_{k,\Omega}$ stand for the semi-norm and the norm in the Sobolev space $H^k(\Omega)$, *i.e.*,

$$| v |^2_{k,\Omega} = \sum_{|q|=k} \int_\Omega (D^q_x v)^2 d\mathbf{x}\,, \qquad \| v \|^2_{k,\Omega} = \|v\|^2_{0,\Omega} + \sum_{l=1}^{k} |v|^2_{l,\Omega}\,,$$

where

$$D^q_x v := \partial^{|q|} v / \partial x_1^{q_1} \partial x_2^{q_2} \partial x_3^{q_3}, \qquad q = (q_1, q_2, q_3), \qquad q_k \geq 0, \qquad |q| = q_1 + q_2 + q_3\,;$$

$\overset{\circ}{H}{}^1(\Omega) := (v \in H^1(\Omega) : v|_{\partial\Omega} = 0)$ is the subspace of functions from $H^1(\Omega)$ vanishing on the boundary $\partial\Omega$. $P_k(x)$ is the Legendre's polynomial of degree $k \geq 1$ for the interval (-1,1). Relationship $\mathbf{v} \leftrightarrow v$ implies that $\mathbf{v}$ is the vector representation of a finite element function v in a chosen basis.

2. Discretization by the spectral elements

As a model, we consider Dirichlet problem: find $u \in \overset{\circ}{H}{}^1(\Omega)$ satisfying the identity

$$a_\Omega(u, v) := \int_\Omega \varrho(x) \nabla u \cdot \nabla v \, d\mathbf{x} = (f, v)_\Omega\,, \qquad \forall\, v \in \overset{\circ}{H}{}^1(\Omega)\,. \tag{2.1}$$

For simplicity, it is assumed that Ω coincides with the computational domain, *i.e.,* it is the domain of an assemblage of geometrically compatible and in general curvilinear finite elements occupying domains τ_r, *i.e.,*

$$\overline{\Omega} = \cup_{r=1}^{\mathcal{R}} \overline{\tau}_r\,.$$

Finite elements and their domains τ_r are specified by nondegenerate mappings $\mathbf{x} = \mathcal{X}^{(r)}(\mathbf{y}) : \overline{\tau}_0 \to \overline{\tau}_r$ with positive Jacobian's, and it is required that these mappings satisfy the conditions, called the *generalized conditions of the angular (shape) quasiuniformity.* If the mappings are trilinear, *i.e.,* elements have straight edges, these conditions are equivalent to the well known conditions of shape regularity, see, *e.g.,* [7]. In a more general case, they are equivalent to the following ones, see [12]. Suppose, each mapping is represented as a superposition of two nondegenerate mappings $\mathcal{X}^{(r)}(\mathbf{y}) = \widetilde{\mathcal{X}}^{(r)}(\mathcal{Z}^{(r)}(\mathbf{y}))$, where $x = \widetilde{\mathcal{X}}^{(r)}(\mathbf{z}) : \bar{\tau}'_r \to \bar{\tau}_r$ is a nonlinear and $\mathbf{z} = \mathcal{Z}^{(r)}(\mathbf{y}) : \bar{\tau}_0 \to \bar{\tau}'_r$ is an affine or trilinear mapping (*e.g.*, with coinciding vertices of τ_r' and τ_r). Then τ_r' must be shape regular, and for the nonlinear mappings and their inverses the Jacobians and their components must be uniformly bounded.

The coefficient ϱ is accepted to be piece-wise constant and such that $\varrho(\mathbf{x}) = \varrho_r = \text{const}$ for $\mathbf{x} \in \tau_r$.

Let us introduce the reference elements $\mathcal{E}_{\text{sp}}$. The coordinates η_i of the GLL nodes on the segment [-1,1] are defined as the roots of the polynomial $(1-s^2)P_p'(s)$, *i.e.*,

$$(1-\eta_i^2)P_p'(\eta_i) = 0\,, \qquad i = 0, 1, .., p\,. \tag{2.2}$$

The GLC nodes are the extremal points of the Chebyshev polynomials

$$\eta_i = \cos\frac{\pi(p-i)}{p}\,, \qquad i = 0, 1, .., p\,. \tag{2.3}$$

The orthogonal meshes with the nodes

$$\mathbf{x} = \boldsymbol{\eta}_{\boldsymbol{\alpha}} = (\eta_{\alpha_1}, \eta_{\alpha_2}, \eta_{\alpha_3})\,, \qquad \boldsymbol{\alpha} \in \omega = (\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) : 0 \le \alpha_1, \alpha_2, \alpha_3 \le p)\,,$$

having the coordinates (2.2) or (2.3), will be termed *Gaussian* for brevity. For coordinate polynomials of the spectral reference elements, we use notation
$J_{\boldsymbol{\alpha}}(\mathbf{x}) = \mathcal{J}_{\alpha_1}(x_1)\mathcal{J}_{\alpha_2}(x_2)\mathcal{J}_{\alpha_3}(x_3)$, where $\mathcal{J}_i(s)$ is the 1-d polynomial of order p, satisfying the equalities $\mathcal{J}_i(\eta_j) = \delta_{i,j}$, $0 \le j \le p$, where $\delta_{i,j}$ – Kronecker's delta.

Without loss of generality, it is convenient to assume here $p = 2N$. For $i \le N$, the steps $\hbar_i := \eta_i - \eta_{i-1}$ of both Gaussian meshes have the same asymptotic behavior $\hbar_i \asymp i/p^2$. One can define a more general class of meshes, which on the segment [-1,0] satisfy the relationships

$$\eta_0 = -1\,, \quad \eta_i = \eta_{i-1} + \hbar_i\,, \qquad \eta_N = 0\,, \qquad c_1\frac{i^\gamma}{\aleph} \le \hbar_i \le c_2\frac{i^\gamma}{\aleph}\,, \qquad \aleph = \sum_{i=1}^{N} i^\gamma\,, \tag{2.4}$$

with some fixed $c_k > 0$ and $\gamma \ge 0$ and are reproduced on [0,1] by the symmetry. For $\gamma = 0$, we have quasiuniform mesh with $\aleph = N$ and for $\gamma = 1$ – the mesh, which will be termed *pseudospectral*, with $\aleph = N(N+1)/2$. In the particular case of $c_1 = c_2 = 1$, one has for the steps of the pseudospectral mesh $\hbar_i = i/\aleph = 2i/(N^2+N) = \beta\, i/p^2$, where $\beta \in [4, 8]$.

The assemblage of spectral finite elements, associated with a single reference element $\mathcal{E}_{\text{sp}}$ by mappings $\mathcal{X}^{(r)}$, defines the FE space

$$\mathbb{V}(\Omega) = \big(v : v \in C(\overline{\Omega}),\;\; v(\mathcal{X}^{(r)}(\mathbf{y}))|_{\mathbf{y}\in\tau_0} \in \mathcal{Q}_{p,\mathbf{x}} \;\text{ for } r = 1, 2, ..., \mathcal{R}\big)\,, \qquad \mathbb{V}(\Omega) \subset H^1(\Omega)\,,$$

and its subspace $\overset{\circ}{\mathbb{V}}(\Omega) = \mathbb{V}(\Omega) \cap \overset{\circ}{H}{}^1(\Omega)$. We write the system of FE algebraic equations for the problem (2.1), obtained by means of the subspace $\overset{\circ}{\mathbb{V}}(\Omega)$, in the form

$$\mathbf{K}\mathbf{u} = \mathbf{f}\,. \tag{2.5}$$

3. Preconditioners for spectral elements

Consider a structured orthogonal mesh specified by coordinates η_i on cube τ_0. Let $\mathcal{H}(\tau_0)$ be the space of functions continuous on $\overline{\tau}_0$ and belonging to $\mathcal{Q}_{1,\mathbf{x}}$ on each cell.

The introduced stiffness matrices can be efficiently preconditioned by the FE matrices $\boldsymbol{\mathcal{A}}_{\rm sp}, \boldsymbol{\mathcal{A}}_{\rm p/s}$, generated by quadratic form a_{τ_0} on the spaces $\mathcal{H}(\tau_0)$ corresponding to the Gaussian and the pseudospectral meshes, respectively. For another preconditioner it can be considered the simpler matrix

$$\mathbb{A}_\hbar = \boldsymbol{\Delta}_\hbar \otimes \mathbb{D}_\hbar \otimes \mathbb{D}_\hbar + \mathbb{D}_\hbar \otimes \boldsymbol{\Delta}_\hbar \otimes \mathbb{D}_\hbar + \mathbb{D}_\hbar \otimes \mathbb{D}_\hbar \otimes \boldsymbol{\Delta}_\hbar\,, \tag{3.6}$$

constructed in terms of matrices $\boldsymbol{\Delta}_\hbar, \mathbb{D}_\hbar$ defined for the 1-d case as follows. Namely, $\mathbb{D}_\hbar$ is the diagonal matrix

$$\mathbb{D}_\hbar = \text{diag}\,[\widetilde{h}_i = \frac{1}{2}(\hbar_i + \hbar_{i+1})]_{i=0}^p\,, \qquad \widetilde{h}_i = 0 \;\text{ for }\; i = 0, p+1\,, \tag{3.7}$$

and $\boldsymbol{\Delta}_\hbar$ is the FE matrix, induced by the bilinear form $(v', w')_{(-1,1)}$ on the space $\mathcal{H}(-1,1)$ of continuous and piece-wise linear on the mesh η_i:

$$\begin{array}{lll} (\boldsymbol{\Delta}_\hbar\,\mathbf{u})|_{i=0} & = -\frac{1}{\hbar_1}(u_1 - u_0)\,, & (\boldsymbol{\Delta}_\hbar\,\mathbf{u})|_{i=p} = \frac{1}{\hbar_p}(u_p - u_{p-1})\,, \\ (\boldsymbol{\Delta}_\hbar\,\mathbf{u})|_{i} & = -\frac{1}{\hbar_i}u_{i-1} + (\frac{1}{\hbar_i} + \frac{1}{\hbar_{i+1}})u_i - \frac{1}{\hbar_{i+1}}u_{i+1}\,, & i = 1, 2, .., p-1\,. \end{array} \tag{3.8}$$

Lemma 3.1. *Let for the same p, matrices $\boldsymbol{\mathcal{A}}_{\rm sp}$ and $\boldsymbol{\mathcal{A}}_{\rm p/s}$ be obtained on the Gaussian mesh and on the pseudospectral mesh at $\gamma = 1$, respectively, whereas $\mathbb{A}_\hbar$ be obtained on either of these meshes. Then they are spectrally equivalent to the stiffness matrix $\mathbf{A}_{\rm sp}$ of the reference element $\mathcal{E}_{\rm sp}$,* i.e.,

$$\mathbb{A}_\hbar, \boldsymbol{\mathcal{A}}_{\rm p/s}, \boldsymbol{\mathcal{A}}_{\rm sp} \prec \mathbf{A}_{\rm sp} \prec \boldsymbol{\mathcal{A}}_{\rm sp}, \boldsymbol{\mathcal{A}}_{\rm p/s}, \mathbb{A}_\hbar \tag{3.9}$$

uniformly in p. Under the same conditions similar inequalities

$$\mathbb{M}_\hbar, \boldsymbol{\mathcal{M}}_{\rm p/s}, \boldsymbol{\mathcal{M}}_{\rm sp} \prec \mathbf{M}_{\rm sp} \prec \boldsymbol{\mathcal{M}}_{\rm sp}, \boldsymbol{\mathcal{M}}_{\rm p/s}, \mathbb{M}_\hbar \tag{3.10}$$

hold for the mass matrix $\mathbf{M}_{\rm sp}$ of the spectral element, its FE preconditioners $\boldsymbol{\mathcal{M}}_{\rm p/s}, \boldsymbol{\mathcal{M}}_{\rm sp}$ obtained with the use of the space $\mathcal{H}(\tau_0)$, and $\mathbb{M}_\hbar = \mathbb{D}_\hbar \otimes \mathbb{D}_\hbar \otimes \mathbb{D}_\hbar$.

4. Domain decomposition preconditioner-solver

At designing the *DD* solver for (2.5), each p-element is treated as a subdomain of decomposition. It is natural to distinguish *internal, face, edge* and *vertex* degrees of freedom in the FE assemblage and respectively decompose the vector space V of the unknowns as

$$V = V_I \oplus V_F \oplus V_E \oplus V_V\,.$$

DD solvers or their parts are also often based on the decompositions

$$V = V_I \oplus V_F \oplus V_W\,, \qquad V = V_I \oplus V_B\,,$$

where $V_B = V_F \oplus V_E \oplus V_V$ and $V_W = V_E \oplus V_V$ are the subspaces of the *interelement boundary* and *wire basket* degrees of freedom. According to these subspaces, the finite element stiffness matrix may be represented in the block forms

$$\mathbf{K} = \begin{pmatrix} \mathbf{K}_I & \mathbf{K}_{IB} \\ \mathbf{K}_{BI} & \mathbf{K}_B \end{pmatrix} = \begin{pmatrix} \mathbf{K}_I & \mathbf{K}_{IF} & \mathbf{K}_{IW} \\ \mathbf{K}_{FI} & \mathbf{K}_F & \mathbf{K}_{FW} \\ \mathbf{K}_{WI} & \mathbf{K}_{WF} & \mathbf{K}_{WW} \end{pmatrix} = \begin{pmatrix} \mathbf{K}_I & \mathbf{K}_{IF} & \mathbf{K}_{IE} & \mathbf{K}_{IV} \\ \mathbf{K}_{FI} & \mathbf{K}_F & \mathbf{K}_{FE} & \mathbf{K}_{FV} \\ \mathbf{K}_{EI} & \mathbf{K}_{EF} & \mathbf{K}_E & \mathbf{K}_{EV} \\ \mathbf{K}_{VI} & \mathbf{K}_{VF} & \mathbf{K}_{VE} & \mathbf{K}_V \end{pmatrix}. \tag{4.11}$$

For the corresponding spaces of the FE functions, we use similar notations with V replaced by $\mathbb{V}$.

The restrictions of the introduced above spaces to the finite elements τ_r are supplied with an additional upper index r, *e.g.,* $\mathbb{V}_B^{(r)}$ denotes the subspace spanned by the boundary coordinate functions of a finite element τ_r with $r = 0$ reserved for the reference cube. Similarly, $\mathbf{K}_F^{(r)}$ is the block of the stiffness matrix of an element τ_r, generated by the face coordinate functions. The spaces $V^{(r)}$ and $\mathbb{V}^{(r)}$ for the reference element will be denoted U and $\mathcal{U} = \mathcal{Q}_{p,\mathbf{x}}$, respectively, with the same indexation for subspaces.

We will consider the DD preconditioner-solver $\boldsymbol{\mathcal{K}}$ for the matrix $\mathbf{K}$ of the form

$$\boldsymbol{\mathcal{K}}^{-1} = \overline{\boldsymbol{\mathcal{K}}}_I^{\,+} + \mathbf{P}_{V_B\to V}\boldsymbol{\mathcal{S}}_B^{-1}\mathbf{P}_{V_B\to V}^\top ,$$
$$(4.12)$$
$$\boldsymbol{\mathcal{S}}_B^{-1} = \overline{\boldsymbol{\mathcal{S}}}_F^{\,+} + \mathbf{P}_{V_W\to V_B}(\boldsymbol{\mathcal{S}}_W^B)^{-1}\mathbf{P}_{V_W\to V_B}^\top ,$$

defined by means of the three preconditioners-solvers: $\boldsymbol{\mathcal{K}}_I^{\,+}$ — for the internal Dirichlet problems on finite elements, $\boldsymbol{\mathcal{S}}_F^{\,+}$ — for the internal problems on faces of finite elements, $(\boldsymbol{\mathcal{S}}_W^B)^{-1}$ — for the wire basket subproblem and two prolongation matrices: $\mathbf{P}_{V_B\to V}$ — from the interelement boundary onto the whole computational domain $\overline{\Omega}$, $\mathbf{P}_{V_W\to V_B}$ — from the wire basket onto the interelement boundary. These components of the domain decomposition preconditioner-solver are defined in the following sections.

5. Preconditioner-solver for the internal Dirichlet problems

In order to obtain fast preconditioner-solver $\boldsymbol{\mathcal{K}}_I^{\,+}$ it is sufficient to design fast preconditioner-solver for the internal stiffness matrices $\mathbf{A}_{I,\mathrm{sp}}$ of spectral elements. Multiresolution wavelet solver for $\mathbf{A}_{I,\mathrm{sp}}$ is derived in [22], we describe it here briefly.

Consider the cube τ_0 subdivided by the cubic mesh of the size $\hbar = 1/p$. Only for convenience and without loss of generality assume, that $p = 2N,\ N = 2^{\ell_0-1}$. For each $l = 1, 2, ..., \ell_0$, one can introduce the uniform mesh $x_i^l,\ i = 0, 1, 2, .., 2N_l,\ N_l = 2^{l-1},\ x_0 = -1,\ x_{2N_l} = 1$ of the size $\hbar_l = 2^{1-l}$ and the space $\mathcal{V}_l(-1, 1)$ of the continuous on (-1,1) piece-wise linear functions, vanishing at the ends of this interval. The dimension of $\mathcal{V}_l(-1, 1)$ is $\mathcal{N}_l = p_l - 1 = 2^l - 1$ with $p_{\ell_0} = p$. Let $\phi_i^l \in \mathcal{V}_l(-1, 1)$ be the the nodal basis function for the node x_i^l, so that $\phi_i^l(x_j^l) = \delta_{i,j}$ and $\mathcal{V}_l(-1, 1) = \mathrm{span}\left\{\phi_i^l\right\}_{i=1}^{p_l-1}$. This basis induces the Gram matrices

$$\boldsymbol{\Delta}_l = \hbar_l\left(\langle(\phi_i^l)', (\phi_j^l)'\rangle_{\omega=1}\right)_{i,j=1}^{p_l-1} , \quad \boldsymbol{\mathcal{M}}_l = \hbar_l^{-1}\left(\langle\phi_i^l, \phi_j^l\rangle_{\omega=\phi}\right)_{i,j=1}^{p_l-1} , \qquad \langle v, u\rangle_\omega := \int_{-1}^{1}\omega^2 v\,u\,dx , \tag{5.13}$$

where

$$\phi(x) = \begin{cases} 1 + x , & x \in [-1, 0] , \\ 1 - x , & x \in [0, 1] , \end{cases}$$

The representation of each $\mathcal{V}_l$ by the direct sum $\mathcal{V}_l = \mathcal{V}_{l-1} \oplus \mathcal{W}_l$ results in the decomposition

$$\mathcal{V} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus ... \oplus \mathcal{W}_{\ell_0}$$

with the notations $\mathcal{V} = \mathcal{V}_{\ell_0}$ and $\mathcal{W}_1 = \mathcal{V}_1$. Let $\{\psi^l_k\}^{p_{l-1},\ell_0}_{k,l=1}$ denote the *multiscale wavelet basis*, composed of some *single scale bases* $\{\psi^l_k\}^{p_{l-1}}_{k=1}$ in the spaces $\mathcal{W}_l$, so that $\mathcal{W}_l = \operatorname{span}\{\psi^l_k\}^{p_{l-1}}_{k=1}$. The multiscale wavelet basis in $\mathcal{V}$, if it is stable in the norms induced by the scalar products (5.13), allows to define 1-d multilevel preconditioners, which in turn lead to the multidimentional tensor product multilevel preconditioner. Before formulating the result, we introduce additional notations. The basis $\{\psi^l_k\}^{p_{l-1},\ell_0}_{k,l=1}$ induces the matrices

$$\boldsymbol{\Delta}_{\mathrm{wlet}} = \left(\langle(\psi^k_i)', (\psi^l_j)'\rangle_1\right)^{p_{l-1};\ell_0}_{i,j=1;\,k,l=1}\,, \qquad \boldsymbol{\mathcal{M}}_{\mathrm{wlet}} = \left(\langle\psi^k_i, \psi^l_j\rangle_\phi\right)^{p_{l-1};\ell_0}_{i,j=1;\,k,l=1}\,, \tag{5.14}$$

$$\mathbb{D}_1 = \operatorname{diag}\left[\langle(\psi^l_i)', (\psi^l_i)',\rangle_1\right]^{p_{l-1},\ell_0}_{i,l=1}\,, \qquad \mathbb{D}_0 = \operatorname{diag}\left[\langle\psi^l_i, \psi^l_i,\rangle_\phi\right]^{p_{l-1},\ell_0}_{i,l=1}\,.$$

By $\mathbf{Q}$ is denoted the transformation matrix from the multiscale wavelet basis to the finite element basis $\{\phi^{l_0}_k\}^{p-1}_{k=1}$. If $\mathbf{v}_{\mathrm{wlet}}$ and $\mathbf{v}$ are the vectors of the coefficients of a function from $\mathcal{V}(0,1)$ in these two bases, respectively, then $\mathbf{v} = \mathbf{Q}^\top\,\mathbf{v}_{\mathrm{wlet}}$.

Theorem 5.1. *There exist multiscale wavelet bases* $\{\psi^l_k\}^{p_{l-1},\ell_0}_{k,l=1}$ *such that the matrices* $\boldsymbol{\Delta}^{-1}_{\ell_0}$ *and* $\boldsymbol{\mathcal{M}}^{-1}_{\ell_0}$ *are simultaneously spectrally equivalent to the matrices* $\mathbf{Q}^\top\mathbb{D}_1^{-1}\mathbf{Q}$ *and* $\mathbf{Q}^\top\mathbb{D}_0^{-1}\mathbf{Q}$, *respectively, uniformly in* p. *Besides, the matrix-vector multiplications* $\mathbf{Q}\,\mathbf{v}_{\mathrm{wlet}}$ *and* $\mathbf{Q}^\top\,\mathbf{v}$ *require* $\mathcal{O}(p)$ *arithmetic operations.*

Theorem 5.2. *Let* $\mathbf{C} = p^{-4}\,\mathbb{D}_\hbar^{-1/2}\otimes\mathbb{D}_\hbar^{-1/2}\otimes\mathbb{D}_\hbar^{-1/2}$, $\boldsymbol{\mathcal{B}}_{I,\mathrm{sp}} = \mathbf{C}\,\mathbb{B}_{I,\mathrm{sp}}\mathbf{C}$ *and*

$$\mathbb{B}^{-1}_{I,\mathrm{sp}} = (\mathbf{Q}^\top\otimes\mathbf{Q}^\top\otimes\mathbf{Q}^\top)[\mathbb{D}_0\otimes\mathbb{D}_0\otimes\mathbb{D}_1+\mathbb{D}_0\otimes\mathbb{D}_1\otimes\mathbb{D}_0+\mathbb{D}_0\otimes\mathbb{D}_0\otimes\mathbb{D}_1]^{-1}(\mathbf{Q}\otimes\mathbf{Q}\otimes\mathbf{Q}) \tag{5.15}$$

then $\boldsymbol{\mathcal{B}}_{I,\mathrm{sp}} \asymp \mathbf{A}_{I,\mathrm{sp}}$. *The arithmetical cost of the operation* $\boldsymbol{\mathcal{B}}^{-1}_{I,\mathrm{sp}}\mathbf{v}$ *for any* $\mathbf{v}\in U_I$ *is* $\mathcal{O}(p^3)$.

The preconditioner-solver for the internal Dirichlet problems on finite elements has the block diagonal form

$$\overline{\boldsymbol{\mathcal{K}}}^{\,+}_I := \begin{pmatrix} \boldsymbol{\mathcal{K}}^{-1}_I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \qquad \text{where} \qquad \boldsymbol{\mathcal{K}}_I = \operatorname{diag}\left[h_1\varrho_1\boldsymbol{\mathcal{B}}_{I,\mathrm{sp}}, h_2\varrho_2\boldsymbol{\mathcal{B}}_{I,\mathrm{sp}}, \dots, h_{\mathcal{R}}\varrho_{\mathcal{R}}\boldsymbol{\mathcal{B}}_{I,\mathrm{sp}}\right]$$

The value h_r is the characteristic size of an element, figuring in the generalized conditions of the angular quasiuniformity. It can be set equal to the arithmetic mean of the inscribed and circumscribed spheres for τ'_r.

Each block $h_r\varrho_r\boldsymbol{\mathcal{B}}_{I,\mathrm{sp}}$ corresponds to one block $\mathbf{K}^{(r)}_I$ in the block $\mathbf{K}_I$ for internal unknowns

$$\mathbf{K}_I = \operatorname{diag}\left[\mathbf{K}^{(1)}_I, \mathbf{K}^{(2)}_I, ..., \mathbf{K}^{(\mathcal{R})}_I\right]$$

of the FE stiffness matrix $\mathbf{K}$.

6. Multiresolution wavelet solver for faces

Another important problem in optimization of DD algorithms for spectral discretizations is the development of fast solvers for the internal problems on faces. As it is known, see, *e.g.,* [27], [19], in the wire basket algorithms it is reduced to the preconditioning of the matrix of the quadratic form ${}_{00}|\cdot|^2_{1/2,F_0}$ on the subspace of polynomials $\mathring{\mathcal{Q}}_{p,\mathbf{x}}$ of two variables $\mathbf{x} = (x_1, x_2)$, vanishing on the boundary of F_0. This quadratic form is the square of the norm in the space $H_{00}^{1/2}(F_0)$, whereas $F_0 = (-1, 1) \times (-1, 1)$ represents a typical face of the 3-d reference cube. As shown in [10] and [23], one of the characterizations of this norm is

$$ {}_{00}| \, v \, |^2_{1/2,F_0} = | \, v \, |^2_{1/2,F_0} + \int_{F_0} \frac{|v(\mathbf{x})|^2}{\mathrm{dist}\,[\mathbf{x}, \partial F_0]} \, d\mathbf{x} $$

Theorem 6.1. *Let $d_{0,i}, d_{1,i}$ be diagonal entries of matrices $\mathbb{D}_0, \mathbb{D}_1$, respectively, and $\mathbb{D}_{1/2}$ be the diagonal $(p-1)^2 \times (p-1)^2$ matrix with the entries on the main diagonal*

$$ d_{i,j}^{(1/2)} = d_{0,i} d_{0,j} \sqrt{\frac{d_{1,i}}{d_{0,i}} + \frac{d_{1,j}}{d_{0,j}}} \, . $$

Let also

$$ \boldsymbol{\mathcal{S}}_0 = \mathbf{C} \, \mathbb{S}_0 \, \mathbf{C} \, , \qquad \mathbb{S}_0^{-1} = (\mathbf{Q}^\top \otimes \mathbf{Q}^\top) \, \mathbb{D}_{1/2}^{-1} \, (\mathbf{Q} \otimes \mathbf{Q}) \, . $$

Then for all $v \in \overset{\circ}{\mathcal{Q}}_{p,\mathbf{x}}$ and vectors $\mathbf{v}$, representing v in the basis $\overset{\circ}{\mathcal{M}}_{2,p}$, the norms ${}_{00}| \, v \, |_{1/2,F_0}$ and $||\mathbf{v}||_{\boldsymbol{\mathcal{S}}_0}$ are equivalent uniformly in p.

The block diagonal preconditioner-solver for the internal problems on faces of finite elements

$$ \overline{\boldsymbol{\mathcal{S}}}{}_F^{+} = \begin{pmatrix} \boldsymbol{\mathcal{S}}_F^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} , \qquad \text{where} \qquad \boldsymbol{\mathcal{S}}_F = \mathrm{diag}\,[\kappa_1 \boldsymbol{\mathcal{S}}_0, \kappa_2 \boldsymbol{\mathcal{S}}_0, \ldots, \kappa_Q \boldsymbol{\mathcal{S}}_0] \, , \tag{6.16} $$

where Q is the number of different faces $F_k \in \Omega$ of the FE discretizations, $\kappa_k = (h_{r_1(k)} \varrho_{r_1(k)} + h_{r_2(k)} \varrho_{r_2(k)})$, and $r_1(k)$, $r_2(k)$ are the numbers of two elements $\overline{\tau}_{r_1(k)}$ and $\overline{\tau}_{r_2(k)}$ sharing the face F_k.

The matrix $\boldsymbol{\mathcal{S}}_F$ is the preconditioner for the $Q(p-1)^2 \times Q(p-1)^2$ block $\mathbf{S}_F$ of the Schur complement

$$ \mathbf{S}_B = \mathbf{K}_B - \mathbf{K}_{BI} \mathbf{K}_I^{-1} \mathbf{K}_{IB} \, , \qquad \mathbf{S}_B = \begin{pmatrix} \mathbf{S}_F & \mathbf{S}_{FW} \\ \mathbf{S}_{WF} & \mathbf{S}_W \end{pmatrix} , $$

which for the reference element is denoted $\mathbb{S}_B$. Although $\mathbf{S}_F$ is not a block diagonal matrix, obviously, it can be represented in the block form with each $(p-1)^2 \times (p-1)^2$ block on the diagonal related to one face.

Theorem 6.2. *The following inequality*

$$ \underline{\gamma}_B \boldsymbol{\mathcal{S}}_B \le \mathbf{S}_B \le \overline{\gamma}_B \boldsymbol{\mathcal{S}}_B \, . \tag{6.17} $$

is true with $\underline{\gamma}_B \ge \underline{c} 1/(1 + \log p)^2$, $\overline{\gamma}_B \le \overline{c}$ and positive constants $\underline{c}, \overline{c}$ depending only on the generalized shape regularity conditions.

7. Prolongation from the interelement boundary

The prolongation $\mathbf{P}_{V_B\to V}$ brings usually the major contribution to the computational cost. For the reason that we have an efficient preconditioner-solver for the internal problems on finite elements, this prolongation can be efficiently completed by means of inexact iterative procedures applied in parallel element wise. We present the simplest variant of such procedure, which adds an extra factor $\log p$ in the estimate of computational work.

The prolongation matrix $\mathbf{P}_{V_B\to V}$ is defined in such a way, that its restriction to each element is $\mathbf{P}^{(r)}_{V_B^{(r)}\to V^{(r)}} = \mathbb{P}_{U_B\to U}$ and $\mathbb{P}_{U_B\to U}$ is the master prolongation matrix, which is used for the prolongation inside any finite element of the discretization. The master prolongation matrix is obtained for the reference element in the following way. Let $\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{A}}_{\rm sp}$ or $\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{A}}_{\rm p/s}$ and $\mathbb{B}_I$ is another preconditioner for the internal block $\mathbf{A}_I$ of the reference element stiffness matrix, possessing a fast solver. To any $\mathbf{v}_B \in U_B$, one can relate the vector $\overline{\mathbf{v}}_B \in U_B$, which entries are equal to the mean value of the corresponding finite element function $v_B \leftrightarrow \mathbf{v}_B$ on the boundary $\partial\tau_0$, and the vector $\widetilde{\mathbf{v}}_B := \mathbf{v}_B - \overline{\mathbf{v}}_B$. By $\overline{\mathbf{v}} \in U$ is denoted the prolongation of $\overline{\mathbf{v}}_B$ by the constant. The prolongation $\mathbf{u} = \mathbb{P}_{U_B\to U}\mathbf{v}_B$ is the sum of two vectors

$$\mathbf{u} = \overline{\mathbf{v}} + \widetilde{\mathbf{u}}\,, \qquad \text{where} \qquad \widetilde{\mathbf{u}} = (\widetilde{\mathbf{u}}_I^\top, \widetilde{\mathbf{v}}_B^\top)^\top\,,$$

and the subvector $\widetilde{\mathbf{u}}_I = \mathbf{w}_I^{k_0}$ is produced for some fixed number k_0 of the iterations

$$\mathbf{w}_I^{k+1} = \mathbf{w}_I^k - \sigma_{k+1}\mathbb{B}_I^{-1}(\boldsymbol{\mathcal{A}}_I\mathbf{w}_I^k - \boldsymbol{\mathcal{A}}_{IB}\widetilde{\mathbf{v}}_B)\,, \qquad \mathbf{w}_I^0 = \mathbf{0}\,, \tag{7.18}$$

with Chebyshev iteration parameters σ_k. Obviously, the order of k_0 will not increase, if to replace $\boldsymbol{\mathcal{A}}_I, \boldsymbol{\mathcal{A}}_{IB}$ by the respective blocks of the reference element stiffness matrix $\mathbf{A}$ or the preconditioner $\mathbb{A}_\hbar$. However, in general case the multiplications by $\mathbf{A}_I, \mathbf{A}_{IB}$ can be much more expensive, than for two other choices. Note, that $\overline{\mathbf{v}}_B$ can be calculated by means of the quadratures, related to the reference element.

Lemma 7.1. *Suppose $\underline{\gamma}_I\mathbb{B}_I \le \boldsymbol{\mathcal{A}}_I \le \overline{\gamma}_I\mathbb{B}_I$ with positive $\underline{\gamma}_I, \overline{\gamma}_I$. Then at*

$$k_0 \ge c(1+\log p)/(\log \rho^{-1})\,,$$

where $\rho = (1-\theta)/(1+\theta)$, $\theta = \sqrt{\underline{\gamma}_I/\overline{\gamma}_I}$, the inequality

$$\|\mathbb{P}_{U_B\to U}\mathbf{v}_B\|_{\,\mathbf{A}} \le c_{\mathbb{P},0}\|\mathbf{v}_B\|_{\,\mathbb{S}_B} \tag{7.19}$$

holds with the constant $c_{\mathbb{P},0}$ independent of p and $\mathbb{S}_B = \mathbf{A}_B - \mathbf{A}_{B,I}\mathbf{A}_I^{-1}\mathbf{A}_{I,B}$.

Let us note that the inequality (7.19) is equivalent to

$$|u|_{1,\tau_0} \prec c_{\mathbb{P},0}|v_B|_{1/2,\partial\tau_0} \tag{7.20}$$

where $u \leftrightarrow \mathbf{u}$ and $v_B \leftrightarrow \mathbf{v}_B$. If $\mathbf{v}_B$ is a constant vector, then $\widetilde{\mathbf{v}}_B = \mathbf{0}$, $\mathbf{A}_{IB}\widetilde{\mathbf{v}}_B = \mathbf{0}$, and, therefore, $\mathbf{w}_I^k = \mathbf{0}$ for $k \ge 1$, $\widetilde{\mathbf{u}} = \mathbf{0}$, and $\mathbb{P}_{U_B\to U}\mathbf{v}_B = \overline{\mathbf{v}}$. According to Lemma 3.1 and Theorem 5.2, the value of ρ is an absolute constant. Therefore, taking additionally to Lemma 7.1 the conditions of the shape quasiuniformity, we come to the following conclusion.

Corollary 7.1. *If $\mathbb{B}_I = \boldsymbol{\mathcal{B}}_{I,\rm sp}$, and $k_0 \asymp (1+\log p)$. Then*

$$\|\mathbf{P}_{V_B\to V}\mathbf{v}_B\|_{\,\mathbf{K}} \le c_{\mathbb{P}}\|\mathbf{v}_B\|_{\,\mathbf{S}_B}\,, \qquad c_{\mathbb{P}} = cc_{\mathbb{P},0}\,, \tag{7.21}$$

with the constant c, depending only on the conditions of the shape regularity.

8. Wire basket component and prolongation onto the interelement boundary

The preconditioner-solver $\boldsymbol{\mathcal{S}}_W^B$ and the prolongation $\mathbf{P}_{V_W \to V_B}$, were studied in *e.g.,* [30], [8], [6], [27]. We borrow them without changes from [6].

Matrices $\boldsymbol{\mathcal{S}}_W^B$, $\mathbf{P}_{V_W \to V_B}$ are assembled of the scaled standard matrices defined for the reference element. For more definiteness we assume that the nodes $\boldsymbol{\eta}_{\boldsymbol{\alpha}}$ of the reference element are the GLL nodes. We denote by $\omega_W \in \omega$ the subset of $\boldsymbol{\alpha}$, corresponding to the nodes on the wire basket W_0 of τ_0, by $\varkappa_{\boldsymbol{\alpha}}$ – the weights of the quadrature, which is assembled of GLL quadratures applied to each edge, and by $\mathbb{S}_{W_0}$ – the matrix of the quadratic form

$$\mathbf{v}_W^\top \mathbb{S}_{W_0} \mathbf{v}_W = \inf_c \sum_{\boldsymbol{\alpha} \in \omega_W} \varkappa_{\boldsymbol{\alpha}} (v_{\boldsymbol{\alpha}} - c)^2 , \tag{8.22}$$

where $v_{\boldsymbol{\alpha}}$ are the entries of $\mathbf{v}_W$. If $\mathbf{D}_0$ is the diagonal matrix of the quadrature in (8.22) and $\mathbf{z}_1$ contains unity for all entries, then

$$\mathbb{S}_{W_0} = \mathbf{D}_0 - \frac{\mathbf{D}_0 \mathbf{z}_1 (\mathbf{D}_0 \mathbf{z}_1)^\top}{\mathbf{z}_1^\top \mathbf{D}_0 \mathbf{z}_1} = \mathbf{D}_0 - \frac{1}{24} \mathbf{D}_0 \mathbf{z}_1 (\mathbf{D}_0 \mathbf{z}_1)^\top \tag{8.23}$$

and $\boldsymbol{\mathcal{S}}_W^B$ is assembled of the matrices

$$\boldsymbol{\mathcal{S}}_W^{(r)} = h_r \, (1 + \log p) \, \varrho_r \, \mathbb{S}_{W_0} \, .$$

For details of the solving procedure for the systems with the preconditioner $\boldsymbol{\mathcal{S}}_W^B$ of the dimension $\mathcal{O}(\mathcal{R}p) \times \mathcal{O}(\mathcal{R}p)$, we refer to Pavarino/Widlund [27] and Casarin [6], where it is described up to solver for the $\mathcal{O}(\mathcal{R}) \times \mathcal{O}(\mathcal{R})$ subsystem. Each unknown in this subsystem corresponds to one element and is coupled only with the next neighboring elements. If the number of elements $\mathcal{R}$ is not fixed, it is sufficient for our purposes to assume that the arithmetical cost of the pointed out procedure does not exceed $\mathcal{O}(\mathcal{R}p^3)$.

Let F_0 be the representative face of the reference element and $\mathbf{v}_{\partial F_0}$ be the vector with the entries related to the nodes on ∂F_0. By definition, the vector $\mathbf{1}_{F_0}$ contains 1-s for all internal nodes of the face F_0 and 0-s for all nodes of its boundary, whereas vector $\mathbf{v}_{F_0}$ is the continuation of $\mathbf{v}_{\partial F_0}$ by zero entries to all internal nodes of the face $\boldsymbol{\eta}_{\boldsymbol{\alpha}} \in \overline{F}_0$. Let also $\overline{v}$ be the mean value on ∂F_0 of the finite element function $v_{\partial F_0} \leftrightarrow \mathbf{v}_{\partial F_0}$, which, *e.g.,* can be calculated by quadratures. Then the standard matrix $\mathbb{P}_{\partial F_0 \to \overline{F}_0}$ for the prolongation from the boundary ∂F_0 on the whole face $\overline{F}_0$ is defined as

$$\mathbb{P}_{\partial F_0 \to \overline{F}_0} \mathbf{v}_{\partial F_0} = \mathbf{v}_{F_0} + \overline{v} \, \mathbf{1}_{F_0} \, . \tag{8.24}$$

A slightly different prolongation is obtained, if $\overline{v}$ is the mean value on ∂F_0 of the piece-wise linear function with the entries of $\mathbf{v}_{\partial F_0}$ for the nodal values. The prolongation matrix $\mathbf{P}_{V_W \to V_B}$ is defined in such a way that its restriction to each face $F_k \in \Omega$ is $\mathbb{P}_{\partial F_0 \to \overline{F}_0}$.

9. Efficiency of the DD preconditioner-solver, its computational cost and parallel algorithm

In this section we cite the result concerning the generalized condition number of the DD preconditioner $\boldsymbol{\mathcal{K}}$, derived in sections 4–8, as well as arithmetic cost of its components and total computational cost of solving system with $\boldsymbol{\mathcal{K}}$.

Theorem 9.1. *DD preconditioner-solver $\boldsymbol{\mathcal{K}}$ provides the condition number*

$$\mathrm{cond}\,[\boldsymbol{\mathcal{K}}^{-1}\mathbf{K}] \le c(1+\log p)^2,$$

whereas the arithmetical cost of the operation $\boldsymbol{\mathcal{K}}^{-1}\mathbf{f}$ for any $\mathbf{f}$ is $\mathcal{O}(p^3(1+\log p)\mathcal{R})$.

For the proof of this theorem, we refer to [22] and remind below only the numbers of arithmetic operations needed for main procedures related to the preconditioning.

The constructed preconditioner $\boldsymbol{\mathcal{K}}$ is aimed for use in the preconditioned conjugate gradient method for solving system (2.5). If we wish to obtain the solution $\mathbf{u}$ with the accuracy ϵ in the energy norm, then according to Th. 9.1 and the general theory of the iterative methods, it is necessary to make $\mathcal{O}((\log p)\log\epsilon^{-1})$ iterations. Less efficient is the two stage method

$$\mathbf{u}^{k+1} = \mathbf{u}^k - \sigma_0\boldsymbol{\mathcal{K}}^{-1}(\mathbf{K}\mathbf{u}^k - \mathbf{f})\,, \qquad \forall\ \mathbf{u}^0\,, \tag{9.25}$$

with the constant iteration parameter σ_0, which requires $\mathcal{O}((\log p)^2\log\epsilon^{-1})$ iterations. There are no difficulties in parallelization of the matrix-vector multiplications, *e.g.,* $\mathbf{K}\mathbf{u}^k$, so that we concentrate below only on the operation $\mathbf{v} := \boldsymbol{\mathcal{K}}^{-1}\mathbf{d}$, $\mathbf{d} = \mathbf{K}\mathbf{u}^k - \mathbf{f}$ which indeed implies solving the system $\boldsymbol{\mathcal{K}}\mathbf{v} = \mathbf{d}$. Let us remind that in general by writing $\mathbf{w} = \mathbf{B}^{-1}\boldsymbol{\phi}$ we always imply solving the system $\mathbf{B}\mathbf{w} = \boldsymbol{\phi}$.

The operation $\boldsymbol{\mathcal{K}}^{-1}\mathbf{f}$ involves the operations with the following arithmetic costs:

i) The block diagonal preconditioner-solver for the internal Dirichlet problems $\boldsymbol{\mathcal{K}}_I^{-1}$ – $\mathcal{O}(p^3\mathcal{R})$ according to Theorem 5.2 and the definition of $\boldsymbol{\mathcal{K}}_I$.

ii) The block diagonal preconditioner-solver for the internal problems on faces of finite elements $\boldsymbol{\mathcal{S}}_F^{-1}$ – $\mathcal{O}(p^2\mathcal{R})$ according to Theorem 6.1 and (6.16).

iii) The preconditioner-solver $\boldsymbol{\mathcal{S}}_W^B$ related to the wire basket subproblem – $\mathcal{O}(p^3\mathcal{R})$ according to the section 8.

iv) The prolongation $\mathbf{P}_{V_B\to V}$ from the interelement boundary onto the whole computational domain – $\mathcal{O}(p^3(1+\log p)\mathcal{R})$ according to Theorem 5.2 and Corollary 7.1.

v) The prolongation $\mathbf{P}_{V_W\to V_B}$ from the wire basket onto the interelement boundary – $\mathcal{O}(p^2\mathcal{R})$ according to the definition of the prolongation $\mathbf{P}_{V_W\to V_B}$ by means of (8.24).

Solution procedure for the system $\boldsymbol{\mathcal{K}}\mathbf{v} = \mathbf{d}$ can be easy parallelized. As it can be seen from the algorithm presented below, the most time consuming operations can be done in parallel element wise and face wise. In particular, solving the internal problems with the fast multiresolution preconditioner-solver described in section 5 as well as the restrictions of the obtained solution to element boundaries and prolongations from the element boundaries inside the elements are completed in parallel for all elements. Solving face problems with Schur complement preconditioners for faces, restrictions of solutions to the edges and prolongations from the edges onto the element faces (see sections 6, 7) can be done in parallel face wise. Details of such an arrangement of computations for solving the system $\boldsymbol{\mathcal{K}}\mathbf{v}=\mathbf{d}$ are seen from the pseudocode of the algorithm presented below.

for $r = 1, 2, .., \mathcal{R}$ **do**
    $\mathbf{v}_{I_r} := \boldsymbol{\mathcal{K}}_{I_r}^{-1}\mathbf{d}_{I_r}$, parallel element wise solving internal Dirichlet problems
    $\mathbf{d}_{B_r}^{(1)} := \mathbf{P}_{V_{B_r}\to V_{I_r}}^{\top}\mathbf{v}_{I_r}$, parallel element wise restrictions to element boundaries
end for
$\mathbf{d}_B^{(1)} := \biguplus \mathbf{d}_{B_r}^{(1)}$, assembling correction to interface subvector of right part
$\mathbf{d}_B := \mathbf{d}_B + \mathbf{d}_B^{(1)}$, updating interface subvector

for $q = 1, 2, .., \mathcal{Q}$ **do**

$\mathbf{v}_{F_q} := \boldsymbol{\mathcal{S}}_{F_q}^{-1}\mathbf{d}_{F_q}$, parallel face wise solving systems with face Schur complement preconditioners

$\mathbf{d}_{W_q}^{(1)} := \mathbf{P}_{V_{W_q}\to V_{F_q}}^{\top}\mathbf{v}_{F_q}$, face wise restrictions to the boundary of the edge in parallel

end for

$\mathbf{d}_W^{(1)} := \biguplus \mathbf{d}_{W_q}^{(1)}$, assembling correction to current wire basket subvector of right part

$\mathbf{d}_W := \mathbf{d}_W + \mathbf{d}_W^{(1)}$, updating current wire basket subvector of right part

$\mathbf{v}_W := \boldsymbol{\mathcal{S}}_W^{-1}\mathbf{d}_W$, solving system with wire basket Schur complement preconditioner

for $q = 1, 2, .., \mathcal{Q}$ **do**

$\mathbf{v}_{F_q}^{(1)} := \mathbf{P}_{V_{W_q}\to V_{F_q}}\mathbf{v}_{W_q}$, parallel face wise prolongations inside faces from their boundaries

$\mathbf{v}_{F_q} := \mathbf{v}_{F_q} + \mathbf{v}_{F_q}^{(1)}$, parallel face wise updating current face subvector

end for

for $r = 1, 2, .., \mathcal{R}$ **do**

$\mathbf{v}_{I_r}^{(1)} := \mathbf{P}_{V_{B_r}\to V_{I_r}}\mathbf{v}_{B_r}$, parallel element wise prolongations inside elements from their boundaries

$\mathbf{v}_{I_r} := \mathbf{v}_{I_r} + \mathbf{v}_{I_r}^{(1)}$, parallel element wise updating of current internal subvectors in parallel

end for

Set $\mathbf{v}_{I_r}, \mathbf{v}_{F_q}, \mathbf{v}_W$ **for components of v, i.e., set**

$\mathbf{v}^{\top} := (\mathbf{v}_{I_1}^{\top}, .., \mathbf{v}_{I_{\mathcal{R}}}^{\top}, \mathbf{v}_{F_1}^{\top}, .., \mathbf{v}_{F_{\mathcal{Q}}}^{\top}, \mathbf{v}_W^{\top})$.

Deeper parallelization can be easily implemented. It is easy to note that the main procedures involve operations only with a few standard matrices, defined for the reference element. These operations can be additionally parallelized without difficulties. There are also some other options. For instance, for solving the internal problems, one can use the secondary domain decomposition method similar to the presented in [2], in which the domain of the reference element can be decomposed in $\mathcal{O}(((\log p)^3)$ subdomains.

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