ANALYSIS OF THE INFLUENCE OF RANDOM NOISE UPON THE PROPERTIES OF STOCHASTIC OSCILLATORS BY THE MONTE CARLO METHOD USING SUPERCOMPUTERS

Ivanov Aleksandr
ICM&MG SB RAS, Novosibirsk
NUMERICAL METHODS FOR SOLVING THE SDE

\[ dy = f(y)dt + \sigma(y)dw(t) \]  \hspace{1cm} (1)

where \( w(t) \) - standard \( M \)-dimensional Wiener process with independent components, \( w(0) = 0, Ew(t) = 0 \) for all \( t \geq 0 \), \( Ew(t)w(s) = \min(t,s), t \geq 0, s \geq 0 \); \( f(y,t) \) - \( K \)-dimensional demolition function, \( \sigma(y,t) \) - \( K \times M \)-diffusion matrix.

The simplest numerical method for solving SDE systems in the sense of Ito is the generalized explicit Euler method:

\[ y_{n+1} = y_n + hf(y_n) + \sqrt{h}\sigma(y_n)\xi_{n+1} \]  \hspace{1cm} (2)

\( \{\xi_{n+1}\}_{0}^{N-1} \) is a sequence of independent \( M \)-dimensional standard Gaussian vectors with independent components, simulated by formula \( \xi = \sqrt{-2 \ln \alpha_1} \sin(2\pi \alpha_2) \) and \( \xi = \sqrt{-2 \ln \alpha_1} \cos(2\pi \alpha_2) \), where \( \alpha_1, \alpha_2 \) are randomly distributed random numbers on the interval \([0,1]\).
The stochastic nonlinear Van-der-Pol equation with one "noise" coefficient, written in the form of a SDE,

\[ dy_1 = y_2 \, dt, \]
\[ dy_2 = (ay_2 (1 - by_1^2) - \omega^2 y_1) \, dt + \sigma_1 y_1 \, dw(t) + \sigma_2 \, dw(t), \]
\[ y_1(0) = 1, \quad y_2(0) = 0, \]

(3)
describes the oscillations of a nonlinear circuit under the influence of internal noise. In (3), the constants \( a, b, \omega \) determine the velocity of the transition sections in the solution and the length of the "shelves".

The SDE system with internal noises for two Duffing oscillators is written in the form:

\[ dy_1 = y_3 \, dt, \]
\[ dy_2 = y_4 \, dt, \]
\[ dy_3 = \left( -ay_3 - (1 + \cos(2t))y_1 - y_1^3 + c(y_4 - y_3) \right) \, dt + \sigma_1 y_1 \, dw(t), \]
\[ y_3(0) = 0, \]
\[ dy_4 = \left( -0.005y_4 - 9(1 + \cos(2t))y_2 - y_2^3 + c(y_3 - y_4) \right) \, dt + \sigma_2 y_2 \, dw(t), \]
\[ y_4(0) = 0, \]

(4)
the connection between two oscillators occurs when \( c \neq 0 \). Components \((y_1, y_3)\) of the solution of the SDE system (5) correspond to the first oscillator; \((y_2, y_4)\) – to the second.
The Lorentz attractor is given by a SDE system of the form:

\[
\begin{align*}
    dy_1 &= -10(y_1 - y_2)\,dt, \\
    dy_2 &= (-y_2 - y_1y_3 + ry_1)\,dt + \sigma y_1\,dw(t), \\
    dy_3 &= \left(-\frac{8}{3}y_3 + y_1y_2\right)\,dt,
\end{align*}
\]

where the parameter \( r \) is considered as "noisy". Lorentz's differential equations arose as a three-mode discrete approximation in the problem of thermal convection between horizontal planes. Analyzing the popular scientific literature in this direction, it was found that the bifurcation point is \( r \approx 19.4 \).

**NUMERICAL EXPERIMENTS**

Numerical experiments were carried out using the AMIKS program. The complex of programs AMIKS (the abbreviation is formed by the first letters of the surnames of developers in English: Artemiev, Marchenko, Ivanov, Korneev, Smirnov) is a convenient software tool for the numerical analysis of SDE systems of any dimension on a supercomputer.
With AMIKS, it is possible to describe various models of stochastic oscillators and obtain various dynamic characteristics of the oscillating system:

- estimating the mathematical expectation;
- the variance estimate;
- distribution density $p(y_i, t_n)$ of the selected component of the solution $y_i(t)$ for a given grid node $t_n$;
- phase portrait, frequency phase portrait (FPP)*;
- frequency integral curve (FIC)**.

The calculated characteristics for solving the SDE system should show the response of the dynamic system to random perturbations, general patterns in behavior and the maximum possible deviations of the simulated trajectories of the SDE solution.

* The FPP plot for the SDE system is an analogue of the phase portrait for the ODE system.
** The FIC plot for the SDE system is an analogue of the integral curve for the ODE system.

In contrast to the histograms in which statistics are collected from the ensemble of simulated SDE trajectories for a fixed time, the FPP and FIC collects complete information on the entire ensemble at each step throughout the entire integration interval.
ABOUT PARALLEL COMPUTING

- We consider nonlinear equations for which there is no exact solution
- The only constructive method for finding the properties of solutions is numerical analysis
- The simulated trajectories are independent and uniformly distributed over the computational nuclei of a supercomputer
- After the end of the calculation they are collected on the central core-collector
- This approach to parallelizing the problems under consideration together with the method (2) yields almost 100% of the efficiency of the computational algorithm
- About acceleration on a supercomputer is not a question in connection with the fact that along with the increase in computing power it is required to calculate a larger amount of data in comparison with a personal computer in order to achieve satisfactory accuracy
- Parallelization, pseudo-random number simulation and other parallel account details are implemented using the PARMONC library*

Fig 1: $a = 0.5$, $b = 3.5$, $\omega = 2\pi$, $\sigma_1 = 1$, $\sigma_2 = 0$ (fig. 1.1, 1.2, 1.3) and $\sigma_1 = 20$ (fig. 1.4, 1.5, 1.6)

Fig 2: $a = 0.5$, $b = 3.5$, $\omega = 2\pi$, $\sigma_1 = 0$, $\sigma_2 = 1$ (Figs. 2.1, 2.2, 2.3) and $\sigma_2 = 20$ (Figs. 2.4, 2.5, 2.6)
Animation 1: additive noise

Animation 2: multiplicative noise
The estimation of the mathematical expectation loses information about one trajectory with time. The frequency integral curve (FIC) gives complete information on the distribution density of the solution over the whole interval of integration.
TEST 2, DUFFING OSCILLATORS

Fig 2, 5: $\sigma_1=\sigma_2=1$
Fig 3, 6: $\sigma_1=\sigma_2=5$

Fig 6: $a=-0.7, c=0$

Fig 7: $a=-0.7, c=5$
Fig 8: $a=-0.7, c=10$
When carrying out the parametric analysis for the value of the multiplicative noise, it was found that for $\sigma > 0.08$ there appears a violation of the bifurcation transition for some trajectories of the SDE solution (5). The studies were carried out in the simulation of 1000 trajectories by the Monte Carlo method using the FPP plot, but it is not ruled out that in modeling a larger number of trajectories some of them will be in both the positive and negative regions. An erroneous result can be obtained by examining the phase portrait, since the bifurcation violation occurs only at a value $\sigma > 0.3$. The estimation graph of the mathematical expectation is stabilized in the zero region, which makes it impossible to derive any information about the individual SDE solution trajectories (5).
CONCLUSION

Our study elucidates the influence of internal and external noise of different intensities on the nonlinear Van-der-Pol equation (3). A comparison has been carried out between the characters of their effects upon the system. When considering the histogram of the distribution density of the solution, it was found that additive noise contributes to the distance of the vertices, and the multiplicative noise contributes to the approximation of the vertices.

It was found that the multiplicative noise affecting the coupled Duffing oscillators (4) promotes synchronization. However, the increase in multiplicative noise shakes the amplitude of oscillations of even synchronous oscillators. It has been shown that coupled oscillators are more resistant to internal noise than individual oscillators. It was noted that for the concomitant ODE system (4) with different binding strength, different modes of oscillation are excited: from damped to periodic.

For the system of Lorentz equations (5), a threshold value of multiplicative noise was obtained, exceeding which it is possible to obtain a violation of the bifurcation transition of the trajectory in the phase plane from the negative to the positive region.
THANK YOU FOR ATTENTION!